Stability to Localized Modes for a Class of Axisymmetric Magnetohydrodynamic Equilibria *

G. Küppers and H. Tasso

Max-Planck-Institut für Plasmaphysik, Garching bei München, Federal Republic of Germany

(Z. Naturforsch. **27 a**, 23—26 [1972]; received 14 October 1971)

Mercier's criterion is computed numerically for an exact class of axisymmetric MHD equilibria having possibly elliptical cross sections and a flat current profile. The main result is that the toroidal current density and the plasma beta can be increased by one order of magnitude if a) the eccentricity of the magnetic surfaces, b) the poloidal current, and c) the aspect ratio are adequately chosen, and if the ellipses are vertical.

I. Introduction

For axisymmetric MHD equilibria Mercier [1] gives a criterion which is necessary for stability with respect to a certain class of perturbations, the so-called localized modes. Evaluations of Mercier's criterion have recently been made [2, 3]. The cross section of the magnetic surfaces is triangular or elliptical. All these calculations are done by an expansion in the aspect ratio.

Here Mercier's criterion is numerically evaluated without any expansion for a class of algebraic axisymmetric equilibria with nearly homogeneous current distribution and having possibly elliptical near-axis cross sections.

In [4, 5] the same equilibrium is given and for special cases a stability calculation was done [5]. A previous paper [6] gives for a special case an exact expression explicit in the equilibrium parameters. This is used here as a test for the numerical calculations. The disagreement ** between the results of our stability calculation and those in [3] may be due to the aspect ratio expansion or may be due to a difference in the equilibrium or more probably to an error in [3].

II. Equilibrium and Geometry

Because we are investigating a toroidal equilibrium with axial symmetry it is convenient to choose the usual cylindrical coordinate system r, φ, z. From the symmetry it follows that all quantities are independent of φ. The magnetic surfaces form a set of nested toroids centered on the magnetic axis at $r=R$. The equilibrium is governed by the equations

$$\boldsymbol{j} \times \boldsymbol{B} = \nabla p \,, \tag{1}$$

$$\operatorname{rot} \boldsymbol{B} = \boldsymbol{j} \,, \tag{2}$$

$$\operatorname{div} \boldsymbol{B} = 0 \,. \tag{3}$$

p being the plasma pressure, $\boldsymbol{B}$ the magnetic field vector, and $\boldsymbol{j}$ the current density. Solving Eq. (3) one gets

$$\boldsymbol{B} = \nabla\varphi \times \nabla\psi + T\,\nabla\varphi \tag{4}$$

where ψ is proportional to the flux the short way and

$$T = r\,B_\varphi \,.$$

From Eq. (2) one gets after using some vector relations

$$\boldsymbol{j} = r^2 \left(\operatorname{div} \frac{\nabla\psi}{r^2} \right) \nabla\varphi + \nabla T \times \nabla\varphi \,. \tag{5}$$

From Eq. (1) it follows that $p = p(\psi)$ and $T = T(\psi)$, and Eq. (1) together with Eqs. (4) and (5) yields one scalar equation describing the equilibrium:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = -r^2 \frac{\mathrm{d}p(\psi)}{\mathrm{d}\psi} - T \frac{\mathrm{d}}{\mathrm{d}\psi} T(\psi) \,. \tag{6}$$

If

$$T^2 = T_0^{\,2} + 4\,\gamma \frac{|p'|}{2(1+\alpha^2)}\,\psi \,, \quad p' = \frac{dp}{\mathrm{d}\psi} \tag{7}$$

and

$$p = -|p'|\,\psi + p_{\max} \tag{8}$$

** Unlike [3] we find that vertical ellipses are more stable than horizontal ellipses.

Reprint requests to Dr. L. Johannsen, Max-Planck-Institut für Plasmaphysik, *D-8046 Garching* bei München.

* This work was performed under the terms of the agreement on association between the Max-Planck-Institut für Plasmaphysik and EURATOM.

with constant T_0, γ, and p' then a class of exact solutions can easily be found:

$$\psi = \left[z^2 (r^2 - \gamma) + \frac{\alpha^2}{4} (r^2 - R^2)^2 \right] \frac{|p'|}{2(1+\alpha^2)} \tag{9}$$

It is convenient to introduce new coordinates

$$X^1 = \sqrt{\frac{\bar{\psi}}{\alpha^2}}, \quad X^2 = \varphi, 2 \operatorname{cotg} X^3 = \alpha \frac{r^2 - R^2}{z (r^2 - \gamma)^{1/2}} \tag{10}$$

with

$$\bar{\psi} = z^2 (r^2 - \gamma) + \frac{\alpha^2}{4} (r^2 - R^2)^2 \tag{11}$$

from which it follows that

$$r = (R^2 + 2 X^1 \cos X^3)^{1/2}, \quad z = \frac{\alpha X^1 \sin X^3}{(R^2 + 2 X^1 \cos X^3 - \gamma)^{1/2}} . \tag{12}$$

The Jacobian can be calculated from

$$r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z = J_n \, \mathrm{d}X^1 \, \mathrm{d}X^2 \, \mathrm{d}X^3 .$$

We obtain

$$J_n = \frac{\alpha X^1}{(R^2 - \gamma + 2 X^1 \cos X^3)^{1/2}} . \tag{13}$$

From Eq. (9) we can calculate the poloidal magnetic field components.

$$B_r = \frac{2z}{r} (r^2 - \gamma) \frac{|p'|}{2(1+\alpha^2)}, \tag{14}$$

$$B_z = - [2 z^2 + \alpha^2 (r^2 - R^2)] \frac{|p'|}{2(1+\alpha^2)} . \tag{15}$$

The toroidal current j_φ is given by

$$j_\varphi = - r p' - \frac{1}{r} T T' . \tag{16}$$

The poloidal current density is proportional to γ. If γ is positive this current is confining the plasma. From Eq. (14) it follows that B_r vanishes at a certain r in the plasma region, if γ is strong enough. In this case there are two magnetic axis and in some sense there exists a critical value of the plasma β for the equilibrium.

In the case

$$\alpha = \alpha_{\text{circle}} = \left(1 - \frac{\gamma}{R^2}\right)^{1/2} \tag{17}$$

one gets a circular near axis cross section of the magnetic surfaces. In general, one has an elliptic near axis cross section. For $\alpha > \alpha_{\text{circle}}$ "vertical" ellipses are obtained. The equilibrium is characterized by the ratio

$$\frac{B_\varphi \, j_\mathrm{p}}{B_\mathrm{p} \, j_\varphi} = \frac{\gamma}{R^2} \frac{1}{(1+\alpha^2) - (\gamma/R^2)}$$

where B_p and j_p are the poloidal magnetic field and poloidal current respectively.

For values less than 1 the equilibrium is tokamak-like and screw-pinch-like otherwise.

III. Stability to Localized Modes

Mercier's criterion is used in the form given in [1]. It is a necessary criterion for stability. If it is violated the plasma is MHD unstable.

The form is

$$Q = \left\{ 1 \Big/ 4 \oint \left(1 + \frac{T^2}{r^2 B^2}\right) \frac{J \, \mathrm{d}\chi}{r^2} \right\} \left[\frac{\partial}{\partial \psi} \oint \frac{T J \, \mathrm{d}\chi}{r^2} - 2 p' T \oint \frac{J \, \mathrm{d}\chi}{r^2 B^2} \right]^2 + p' \oint \frac{\partial}{\partial \psi} J \, \mathrm{d}\chi - p'^2 \oint \frac{J \, \mathrm{d}\chi}{B^2} > 0, \quad J \, \mathrm{d}\chi = \frac{1+\alpha^2}{\alpha |p'|} \frac{\mathrm{d}X^3}{(R^2 + 2 X^1 \cos X^3 - \gamma)^{1/2}} . \tag{18}$$

The stability calculation was done numerically. First it can be shown numerically that Q is a monotonically increasing function of X^1. This means that if Q becomes negative this happens near the axis.

From Eqs. (7), (9), (10), (14), (15) and (18) it can easily be seen that near the axis Q depends on p' and T only in the following way

$$Q(p', T) = Q(p'/T_0) .$$

Therefore, Q can be calculated in the neighbourhood of the magnetic axis as a function of p'/T_0 for different values of α and γ, and one can determine that value of p'/T_0 for which $Q(p'/T_0) = 0$.

Equation (16) yields

$$j_\varphi = R |p'| \left(1 - \frac{\gamma}{R^2} \frac{1}{(1+\alpha^2)}\right) \tag{19}$$

from which one can calculate that value of j_φ for which Q is equal to zero. This yields the stability diagram in Figure 1.

The results are:

1. For $\gamma = 0$, i. e. the poloidal current is equal to zero, a circular cross section is more stable than an elliptic one in the sense that a circular cross section allows higher values of j_φ.

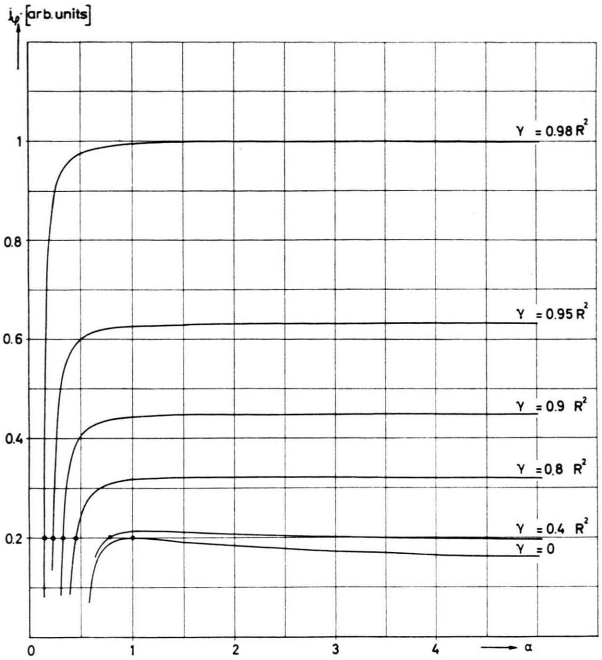

Fig. 1. The limit of the toroidal current j_φ versus α for different values of γ. γ is proportional to the poloidal current density; α denotes for a given γ a certain eccentricity of the elliptic near axis cross sections of the magnetic surfaces. The values of α corresponding to a circular cross section are marked with a point.

2. In the case of non-vanishing poloidal current the stability, i. e. the limit of j_φ, increases with increasing γ. But the maximum value of γ is determined by the equilibrium in such a way that B_r goes to zero identically for a given value of r on the plasma boundary. The second magnetic axis appears at the boundary. This yields

$$\gamma \leqq R^2 - 2\,X^1_{\text{boundary}} \approx R^2(1-2\,\varepsilon) \tag{20}$$

where ε is the inverse aspect ratio. Equations (10) and (11) are used. X^1_{boundary} is defined by

$$p = 0 = p'\,\psi + p_{\max}\,.$$

To provide high values of γ, it follows from Eq. (20) that the torus has to be slim. In other words, a certain inverse aspect ratio ε allows a certain maximum value $\gamma_{\max}(\varepsilon)$ of γ. In the following

$$\gamma = \gamma_{\max} \approx R^2(1-2\,\varepsilon)\,.$$

From Eq. (9) one gets for the ratio A of the minor to the major axis in the case of elliptic cross sections

$$A = \left(\frac{R^2-\gamma}{R^2\,\alpha^2}\right)^{1/2} \approx \frac{1}{\alpha}\,\sqrt{2\,\varepsilon}\,. \tag{21}$$

From the stability diagram it can be seen that for each choice of γ, j_φ reaches practically its maximal value for $\alpha = 1$. This corresponds for a given aspect ratio to an elliptic cross section of the plasma with an eccentricity e

$$e = (1-2\,\varepsilon)^{1/2}\,. \tag{22}$$

Furthermore, it can be seen that for increasing values of γ the ratio $j_\varphi(\alpha=1)$ to $j_\varphi(\alpha=\alpha_{\text{circle}})$ increases from 1 to 5.

From Eqs. (7) and (8) the toroidal plasma β can be calculated:

$$\beta_T = 2\,\frac{p_{\max}}{B_\varphi{}^2} = \frac{S^2\,\varepsilon^2\,\alpha^2}{(1+\alpha^2)\left\{1-\dfrac{\alpha^2}{1+\alpha^2}\,(1-2\,\varepsilon)\,S^2\,\varepsilon^2\right\}} \tag{23}$$

where

$$S = (|p'|/T_0)\,R^3\,.$$

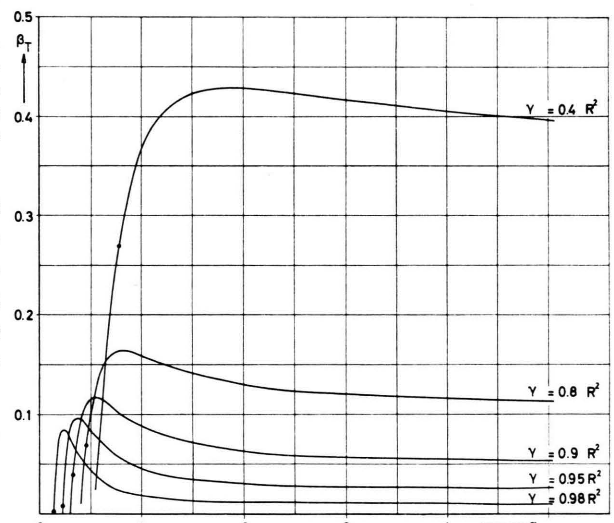

Fig. 2. The limit of the toroidal plasma β versus α for different values of γ. γ is proportional to the poloidal current density; α denotes for a given γ a certain eccentricity of the elliptic near axis cross sections of the magnetic surfaces. The values of α corresponding to a circular cross section are marked with a point.

In Fig. 2 β_T is calculated as a function of α for different values of γ and for values p'/T_0 for which $Q=0$. The result is that for elliptic cross sections β_T becomes greater than for a circular cross section. For high values of γ the relative growth is greater, though absolutely β_T is higher for smaller γ.

IV. Conclusion

It has been shown that in a certain range of α both much higher values of j_φ and much higher values of β_T compared with the corresponding values for circular cross section can be reached, if the ellipses are vertical.

We are grateful to Mrs. PILLSTICKER for some numerical calculations.

[1] C. MERCIER, Nucl. Fusion 1, 47 [1960].
[2] H. LUC, EUR-CEA-FC-569 (Oct. 1970).
[3] G. LAVAL, E. K. MASCHKE, and R. PELLAT, IC/70/110, Trieste Report (Aug. 1970).
[4] W. B. THOMPSON, An Introduction to Plasma Physics, Pergamon Press, London 1962, p. 55.
[5] L. S. SOLOV'EV, J. Exp. Theor. Phys. USSR **26**, 400 [1968].
[6] G. KÜPPERS and H. TASSO, Phys. Letters 35 A, **6**, 410 [1971].

Driftgeschwindigkeit und Beweglichkeit von Elektronen in Silicium bei 4,2 °K in Abhängigkeit vom elektrischen Feld

P. DEIMEL *

Atominstitut der Österreichischen Hochschulen, Wien, Austria

(Z. Naturforsch. **27 a**, 26—30 [1972]; eingegangen am 26. Mai 1971)

Driftvelocity and Mobility of Electrons in Silicon at 4.2 °K in Dependence of Electric Field

The pulse rise times of an n-type silicon surface barrier detector were measured at 4.2 °K. At this temperature the detector was fully depleted even at very low bias and the measured pulse rise times gave direct information about the driftvelocity and the mobility. Instead of $E^{-0.5}$, an $E^{-0.8}$ dependence of the mobility at moderate electric fields was found. At high electric fields agreement exists with theory.

1. Einleitung

Die Beweglichkeit der Ladungsträger in Halbleitern in Abhängigkeit von höheren elektrischen Feldern ist sowohl bei Raumtemperatur als auch bei der Temperatur des flüssigen Stickstoffes mehrfach untersucht worden [1–6]. Hingegen ist die Zahl der Arbeiten, die sich mit dem Verhalten der Driftgeschwindigkeit in hochohmigem Silicium bei 4,2 °K beschäftigen, sehr gering [7].

Ziel dieser Arbeit ist, durch Messung der Anstiegszeit der von α-Teilchen ausgelösten Impulse eines hochohmigen n-Siliciumoberflächensperrschichtzählers bei 4,2 °K Aussagen über Driftgeschwindigkeit und Beweglichkeit und deren Abhängigkeit von einem äußeren elektrischen Feld zu gewinnen.

2. Theorie der Beweglichkeit und Driftgeschwindigkeit

In diesem Abschnitt soll eine Zusammenstellung der bisher bekannten und experimentell bestätigten theoretischen Aussagen über die Beweglichkeit und die Driftgeschwindigkeit und ihre Abhängigkeit von einem äußeren elektrischen Feld erfolgen. Diese Gleichungen sollen auch bei 4,2 °K als Berechnungsgrundlage dienen, wobei die auf diesem Wege gewonnenen numerischen Werte mit den experimentell gewonnenen verglichen werden sollen.

Die Beweglichkeit wird in den meisten Halbleitern prinzipiell durch Streuung an Gitterschwingungen und an ionisierten Störstellen bestimmt. Im Fall der Streuung an Gitterschwingungen kann Wechselwirkung sowohl mit optischen als auch akustischen Phononen erfolgen. Bei niedriger Temperatur ist aber die Streuung durch optische Phononen gegenüber der durch akustische Phononen vernachlässigbar [8, 9], solange die Ladungsträger keine zu hohe Elektronentemperatur besitzen. Da somit im Falle der Gitterschwingung bei Berechnung der Niederfeldbeweglichkeit nur akustische Phononen wechselwirken, ergibt sich eine Abhängigkeit der Niederfeldbeweglichkeit μ_{0T} von der Gittertemperatur T

Sonderdruckanforderungen an Dr. P. DEIMEL, Physik-Department E 18, Techn. Universität München, *D-8046 Garching*, Freisinger Landstraße.

* Nun Physik-Department der Technischen Universität München, Germany.